Relativistic entanglement of quantum states and nonlocality of Einstein-Podolsky-Rosen(EPR) paradox

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Abstract

Relativistic bipartite entangled quantum states is studied to show that Nature doesn't favor nonlocality for massive particles. We found that to an observer (Bob) in a moving frame S', the entangled Bell state shared by Alice and Bob appears as the superposition of the Bell bases in the frame S' due to the requirement of the special relativity. When Alice measures her spin in the positive z-direction at $t = \tau$, it was shown that Bob's spin state is still undetermined in contrast to the non-relativistic case, thus providing a counter example for nonlocality of Einstein-Podolsky-Rosen(EPR) paradox.

Entanglement of bipartite quantum states is of fundamental interest for quantum information processing such as quantum computation [1]– [7], teleportation [8]– [11] and clock synchronization [12]– [14]. How does the entangled quantum states appear to an observer in a different Lorentz frame would be an interesting question, potentially related to the clock

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synchronization problem related to the quantum entanglement. Another unsolved problem, perhaps more important than the above one, is the violation of the local causality in quantum mechanics by measurement process, so called, the Einstein-Podolsky-Rosen(EPR) paradox [15] and the Bell's theorem [16], suggesting the existence of an instantaneous action between distant measurements. This subtle questions still remain to be answered even though there have been several works [17]– [19] relating the relativity, entanglements and the quantum operations.

One of the conceptual barriers for the relativistic treatment of quantum information processing is the difference of the role played by the wave fields and the state vectors in the quantum field theory. In non-relativistic quantum mechanics both the wave function and the state vector in Hilbert space give the probability amplitude which can be used to define conserved positive probability densities or density matrix. On the other hand, in relativistic quantum field theory, the wave fields are not probability amplitude at all, but operators which create or destroy particles in spanned by states defined as containing definite numbers of particles or antiparticles in each normal mode [20]. Moreover, there has been a quandary [17]– [19] whether the quantum states are Lorentz covariant but according to Weinberg [20], the quantum states viewed from different reference frames can be represented by the Lorentz transformation.

Recently, Alsing and Milburn [21] studied the Lorentz invariance of entanglement and showed that the entanglement fidelity of the bipartite state is preserved explicitly. To the best of our knowledge, their work is the first detailed calculation of the relativistic quantum entanglement of bipartite state. However, in their approach, it is not quite clear whether the entanglement is for the quantum state or the quantum fields because they started from the entanglement between the 4-spinors for the Dirac field. In quantum field theory, the role of the field is to make the interaction or the S-matrix satisfying the Lorentz invariance and the cluster decomposition principle. On the other hand, the information of the particle states is contained in the state vectors of the Hilbert space spanned by states containing $0, 1, 2, \cdots,$

particles as in the case of non-relativistic quantum mechanics [20].

In this article, we study the Lorentz transformation properties of entanglement of bipartite quantum states in the Hilbert space and provide the counter example for the nonlocality of the EPR paradox. Throughout the article, we follow Weinberg's notation [20]. A multi-particle state vector is denoted by

$$\Psi_{p_1,\sigma_1;p_2,\sigma_2;\cdots} = a^+(\vec{p_1},\sigma_1)a^+(\vec{p_2},\sigma_2)\cdots\Psi_0, \tag{1}$$

where p_i labels the four-momentum, σ_i is the spin z component, $a^+(\vec{p_i},\sigma_i)$ is the creation operator which adds a particle with momentum $\vec{p_i}$, and spin σ_i, and Ψ_0 is the Lorentz invariant vacuum state. The Lorentz transformation Λ induces a unitary transformation on vectors in the physical Hilbert space

$$\Psi \rightarrow U(\Lambda)\Psi, \tag{2}$$

and the operators U satisfies the composition rule

$$U(\bar{\Lambda})U(\Lambda) = U(\bar{\Lambda}\Lambda), \tag{3}$$

while the creation operator has the following transformation rule [20]

$$U(\Lambda)a^+(\vec{p},\sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\bar{\sigma}}\mathcal{D}^{(j)}_{\bar{\sigma}\sigma}(W(\Lambda,p))a^+(\vec{p_\Lambda},\bar{\sigma}). \tag{4}$$

Here, $W(\Lambda,p)$ is the Wigner's little group element given by

$$W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p), \tag{5}$$

$\mathcal{D}^{(j)}(W)$ the representation of W for spin j, $p^\mu = (\vec{p},p^0)$ and $(\Lambda p)^\mu = (\vec{p_\Lambda},(\Lambda p)^0)$ with $\mu = 1,2,3,0$, and $L(p)$ is the standard Lorentz transformation such that

$$p^\mu = L^\mu_\nu(p)k^\nu, \tag{6}$$

where $k^\nu = (0,0,0,m)$ is the four-momentum taken in the particle's rest frame.

The relativistic momentum-conserved entangled Bell states for spin $\frac{1}{2}$ particles in the rest frame S are defined by

$$\Psi_{00} = \frac{1}{\sqrt{2}}\{a^+(\vec{p}, \frac{1}{2})a^+(-\vec{p}, \frac{1}{2}) + a^+(\vec{p}, -\frac{1}{2})a^+(-\vec{p}, -\frac{1}{2})\}\Psi_0, \tag{7a}$$

$$\Psi_{01} = \frac{1}{\sqrt{2}}\{a^+(\vec{p}, \frac{1}{2})a^+(-\vec{p}, \frac{1}{2}) - a^+(\vec{p}, -\frac{1}{2})a^+(-\vec{p}, -\frac{1}{2})\}\Psi_0, \tag{7b}$$

$$\Psi_{10} = \frac{1}{\sqrt{2}}\{a^+(\vec{p}, \frac{1}{2})a^+(-\vec{p}, -\frac{1}{2}) + a^+(\vec{p}, -\frac{1}{2})a^+(-\vec{p}, \frac{1}{2})\}\Psi_0, \tag{7c}$$

$$\Psi^{11} = \frac{1}{\sqrt{2}}\{a^+(\vec{p}, \frac{1}{2})a^+(-\vec{p}, -\frac{1}{2}) - a^+(\vec{p}, -\frac{1}{2})a^+(-\vec{p}, \frac{1}{2})\}\Psi_0, \tag{7d}$$

where Ψ_0 is the Lorentz invariant vacuum state. It is straightforward to see that the momentum-conserved Bell states $(7a) - (7d)$ have both the space inversion ($\mathcal{P}$) and the time-reversal ($\mathcal{T}$) symmetries.

For an observer in another reference frame S' described by an arbitrary boost Λ, the transformed Bell states are given by

$$\Psi_{ij} \rightarrow U(\Lambda)\Psi_{ij}. \tag{8}$$

For example, from equations (4) and (7a), $U(\Lambda)\Psi_{00}$ becomes

$$\begin{aligned} U(\Lambda)\Psi_{00} = & \frac{1}{\sqrt{2}}\{U(\Lambda)a^+(\vec{p}, \frac{1}{2})U^{-1}(\Lambda)U(\Lambda)a^+(-\vec{p}, \frac{1}{2})U^{-1}(\Lambda) \\ & +U(\Lambda)a^+(\vec{p}, -\frac{1}{2})U^{-1}(\Lambda)U(\Lambda)a^+(-\vec{p}, -\frac{1}{2})U^{-1}(\Lambda)\}U(\Lambda)\Psi_0 \\ = & \frac{1}{\sqrt{2}}\sum_{\sigma,\sigma'}\{\sqrt{\frac{(\Lambda p)^0}{p^0}}\mathcal{D}^{(\frac{1}{2})}_{\sigma\frac{1}{2}}(W(\Lambda, p))\sqrt{\frac{(\Lambda\mathcal{P}p)^0}{(\mathcal{P}p)^0}}\mathcal{D}^{(\frac{1}{2})}_{\sigma'\frac{1}{2}}(W(\Lambda, \mathcal{P}p))a^+(\vec{p}_\Lambda, \sigma)a^+(\vec{p}_\Lambda, \sigma') \\ & +\sqrt{\frac{(\Lambda p)^0}{p^0}}\mathcal{D}^{(\frac{1}{2})}_{\sigma-\frac{1}{2}}(W(\Lambda, p))\sqrt{\frac{(\Lambda\mathcal{P}p)^0}{(\mathcal{P}p)^0}}\mathcal{D}^{(\frac{1}{2})}_{\sigma'-\frac{1}{2}}(W(\Lambda, \mathcal{P}p))a^+(\vec{p}_\Lambda, \sigma)a^+(\vec{p}_\Lambda, \sigma')\}\Psi_0 \end{aligned} \tag{9}$$

and so on. Here $\mathcal{P}$ is the space-inversion operator. For simplicity, we assume that $\vec{p}$ is in z-direction, $\vec{p} = (0, 0, p)$ and the boost Λ is in x-direction. Then, we have

$$L(p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\eta & \sinh\eta \\ 0 & 0 & \sinh\eta & \cosh\eta \end{bmatrix}, \tag{10a}$$

$$L(\mathcal{P}p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\eta & -\sinh\eta \\ 0 & 0 & -\sinh\eta & \cosh\eta \end{bmatrix}, \tag{10b}$$

and

$$\Lambda = \begin{bmatrix} \cosh\omega & 0 & 0 & \sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\omega \end{bmatrix} \tag{10c}$$

where η and ω are the boost in z- and x-directions, respectively. The matrix representation of the Wigner's little group W is given by [21]

$$\mathcal{D}^{(\frac{1}{2})}_{\sigma'\sigma}(W(\Lambda,p)) = \begin{pmatrix} \cos(\frac{\Omega_p}{2}) & -\sin(\frac{\Omega_p}{2}) \\ \sin(\frac{\Omega_p}{2}) & \cos(\frac{\Omega_p}{2}) \end{pmatrix}, \tag{11a}$$

and

$$\mathcal{D}^{(\frac{1}{2})}_{\sigma'\sigma}(W(\Lambda,\mathcal{P}p)) = \begin{pmatrix} \cos(\frac{\Omega_p}{2}) & \sin(\frac{\Omega_p}{2}) \\ -\sin(\frac{\Omega_p}{2}) & \cos(\frac{\Omega_p}{2}) \end{pmatrix}, \tag{11b}$$

where the Wigner angle Ω_p is defined by [21]

$$\tan\Omega_p = \frac{\sinh\eta\sinh\omega}{\cosh\eta + \cosh\omega}. \tag{12}$$

By substituting equations (11a) and (11b) into equation (9), we obtain

$$\begin{aligned} U(\Lambda)\Psi_{00} &= \frac{(\Lambda p)^0}{p^0}\cos\Omega_p\frac{1}{\sqrt{2}}\{a^+(\vec{p},\frac{1}{2})a^+(-\vec{p},\frac{1}{2}) + a^+(\vec{p},-\frac{1}{2})a^+(-\vec{p},-\frac{1}{2})\}\Psi_0 \\ &\quad -\frac{(\Lambda p)^0}{p^0}\sin\Omega_p\frac{1}{\sqrt{2}}\{a^+(\vec{p},\frac{1}{2})a^+(-\vec{p},-\frac{1}{2}) - a^+(\vec{p},-\frac{1}{2})a^+(-\vec{p},\frac{1}{2})\}\Psi_0 \\ &= \frac{(\Lambda p)^0}{p^0}\{\cos\Omega_p\Psi'_{00} - \sin\Omega_p\Psi'_{11}\}, \end{aligned} \tag{13a}$$

where $\Psi'ij$ is the Bell states in the moving frame S' whose momentums are transformed as $\vec{p} \to \vec{p}_\Lambda$, $-\vec{p} \to -\vec{p}_\Lambda$. Likewise, we have

$$U(\Lambda)\Psi_{01} = \frac{(\Lambda p)^0}{p^0}\Psi'_{01}, \tag{13b}$$

$$U(\Lambda)\Psi_{10} = \frac{(\Lambda p)^0}{p^0}\Psi'_{10},, \tag{13c}$$

and

$$U(\Lambda)\Psi_{11} = \frac{(\Lambda p)^0}{p^0}\{\sin\Omega_p\Psi'_{00} + \cos\Omega_p\Psi'_{11}\}. \tag{13d}$$

In the work of Alsing and Milburn [21], they studied the Lorentz transformation of entangled four-spinors of the Dirac field and suggested that an observer S' travelling along the x-direction with constant velocity with respect to an observer S in the rest frame will observe a rotation of the spins in the direction of the boost by an angle Ω_p. However, as we mentioned earlier, the particle states should be represented by the physical state vectors in Hilbert space and not by the quantum fields. Fields are introduced by the requirement that the S-matrix (or the interaction) satisfy Lorentz invariance and cluster decomposition principle. If we regard Ψ'_{ij} as Bell states in the moving frame S', then to an observer in S', the effects of the Lorentz transformation of the bipartite entangled Bell states should appear as the superpositions of Bell states in the frame S'.

The implications could be non trivial. One of the controversies in modern physics is the violation of the local causality of relativistic quantum field theory during the measurement process [22]. This is based on the EPR paradox [15] and the Bell's theorem [16], which suggest the existence of nonlocal instantaneous action between distant measurements. In the following, we investigate whether a supposed nonlocality is a real physical property of the quantum theory, more specially, the result of state collapse description by studying the case of an entangled state shared by Alice and Bob in the relativistic regime. For example, consider Alice in the frame S and Bob in the frame S' (initially coincide with S) moving in the x-direction share entangled pair of atoms whose electrons have opposite momentum prepared at certain time $t = 0$. At time $t = 0$, the entangled state shared by Alice and Bob is assumed to be Ψ^{AB}_{00},

$$\Psi^{AB}_{00} = \Psi_{00}$$

$$= \frac{1}{\sqrt{2}}\{a_A^+(\vec{p},\frac{1}{2})a_B^+(-\vec{p},\frac{1}{2}) + a_A^+(\vec{p},-\frac{1}{2})a_B^+(-\vec{p},-\frac{1}{2})\}\Psi_0. \tag{14}$$

Here A and B denote particles belong to Alice and Bob, respectively. When the reference frame S' where Bob is in, is moving, the Lorentz boost Λ will affect only the Alice's state and as a result the global unitary transformation can be written as

$$U_{AB} = U_A(\Lambda) \otimes I_B, \tag{15}$$

where $U_A(\Lambda)$ is the unitary transformation representing the Lorentz boost upon Alice. Then the quantum state from Bob's point of view is given by

$$\begin{aligned}
U_{AB}(\Lambda)\Psi_{00}^{AB} &= \frac{1}{\sqrt{2}}\sum_\sigma \sqrt{\frac{(\Lambda p)^0}{p^0}}\{\mathcal{D}^{(\frac{1}{2})}_{\sigma\frac{1}{2}}(W(\Lambda,p))a_A^+(\vec{p}_\Lambda,\sigma)a_B^+(-\vec{p},\frac{1}{2}) \\
&\quad +\mathcal{D}^{(\frac{1}{2})}_{\sigma-\frac{1}{2}}(W(\Lambda,p))a_A^+(\vec{p}_\Lambda,\sigma)a_B^+(-\vec{p},-\frac{1}{2})\}\Psi_0 \\
&= \sqrt{\frac{(\Lambda p)^0}{p^0}}[\cos\frac{\Omega_p}{2}\frac{1}{\sqrt{2}}\{a_A^+(\vec{p}_\Lambda,\frac{1}{2})a_B^+(-\vec{p},\frac{1}{2}) + a_A^+(\vec{p}_\Lambda,-\frac{1}{2})a_B^+(-\vec{p},-\frac{1}{2})\} \\
&\quad -\sin\frac{\Omega_p}{2}\frac{1}{\sqrt{2}}\{a_A^+(\vec{p}_\Lambda,\frac{1}{2})a_B^+(-\vec{p},-\frac{1}{2}) - a_A^+(\vec{p}_\Lambda,-\frac{1}{2})a_B^+(-\vec{p},\frac{1}{2})\}]\Psi_0 \\
&= \sqrt{\frac{(\Lambda p)^0}{p^0}}[\cos\frac{\Omega_p}{2}\Psi'^{AB}_{00} - \sin\frac{\Omega_p}{2}\Psi'^{AB}_{11}],
\end{aligned} \tag{16}$$

Now Alice performs the measurement of the spin in the $+z$ direction at time $t = \tau$. Since the Bell state $U_{AB}(\Lambda)\Psi_{00}^{AB}$ viewed from Bob in the frame S' is a linear combination of Ψ'^{AB}_{00} and Ψ'^{AB}_{11}, when Alice measures her spin in the positive z direction, Bob's spin state is still a linear combination of $|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$. This leaves Bob's spin direction undetermined contradicting the EPR paradox. On the other hand, in the non-relativistic quantum mechanics, Bob's quantum state is determined instantaneously as a result of collapse when Alice does her measurement which results in the violation of the Bell inequality or the EPR paradox [23]. One might ask the validity of our result since there are experimental evidences [24]– [26] showing the violation of Bell's inequality which suggests that reality is nonlocal. So far the experiments that test Bell's inequality are done with entangled photons which are massless, not with spin-$\frac{1}{2}$ massive particles. It is interesting to note that the representation of the Wigner's little group W for the massless particle is diagonal [20], i.e.,

$$\mathcal{D}_{\sigma\sigma'}(W) = \exp(i\theta\sigma)\delta_{\sigma'\sigma}, \tag{17}$$

where θ is the angle related to the Lorentz boost Λ. So the form of the entanglement is left invariant even after the Lorentz boost and this will give the similar results as in the case of non-relativistic quantum mechanics. However, for massive particles, it doesn't look like that Nature favors nonlocality suggested by the EPR paradox if one wants to reconcile the principles of quantum mechanics with those of special relativity.

In conclusion, we studied the Lorentz transformation of the bipartite entangled quantum states explicitly and found that to an observer in a moving frame, the Bell states appear as rotations (or linear combination) of the Bell bases in that frame. It turns out that the superposed nature of the Lorentz transformed Bell states could serve as a counter example for the nonlocality of EPR paradox.

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